Optimal control under probability constraint

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Presentation outline

Satellite model

$$\frac{\mathrm{d}r}{\mathrm{d}t} = v \; , \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -\mu \frac{r}{\|r\|^3} + \frac{F}{m}\kappa \; , \tag{1a}$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{T}{g_0 I_{\mathrm{sp}}}\delta \; . \tag{1b}$$

(1a) : 6-dimensional state vector (position r and velocity v).

(1b) : 1-dimensional state vector (mass m including fuel).

κ involves the direction cosines of the thrust and the on-off switch δ of the engine (3 controls), and $\mu, F, T, g_0, I_{\mathrm{sp}}$ are constants.

The deterministic control problem is to drive the satellite from the initial condition at t_{i} to a known **final position** r_{f} **and velocity** v_{f} at t_{f} (given) while **minimizing fuel consumption** $m(t_{\mathrm{i}}) - m(t_{\mathrm{f}})$.

Deterministic optimization problem

Using **equinoctial coordinates** for the position and velocity

$\rightsquigarrow$ state vector $x \in \mathbb{R}^7$,

and **cartesian coordinates** for the thrust of the engine

$\rightsquigarrow$ control vector $u \in \mathbb{R}^3$,

the deterministic optimization problem is written as follows:

$$\min_{u(\cdot)} K\big(x(t_\mathrm{f})\big) \tag{2a}$$

subject to:

$$x(t_\mathrm{i}) = x_\mathrm{i} \ , \quad \overset{\bullet}{x}(t) = f\big(x(t), u(t)\big) \ , \tag{2b}$$

$$\|u(t)\| \leq 1 \quad \forall t \in [t_\mathrm{i}, t_\mathrm{f}] \ , \tag{2c}$$

$$C\big(x(t_\mathrm{f})\big) = 0 \ . \tag{2d}$$

Engine failure

- Sometimes, the engine may **fail to work** when needed: the satellite **drift away** from the deterministic optimal trajectory. After the engine control is recovered, it is not always possible to **drive the satellite to the final target** at t_{f}.
- By **anticipating** such possible failures and by **modifying** the trajectory followed **before** any such failure occurs, one may **increase** the possibility of eventually reaching the target. But such a deviation from the deterministic optimal trajectory results in a **deterioration of the economic performance**.
- The problem is thus to **balance** the **increased probability** of eventually reaching the target despite possible failures against the **expected economic performance**, that is, to **quantify** the price of safety one is ready to pay for.

Stochastic formulation (1)

A failure is modeled using two random variables:

- $\mathbf{t}_{\mathrm{p}}$: random initial time of the failure,
- $\mathbf{t}_{\mathrm{d}}$: random duration of the failure.

For every realization $(t_{\mathrm{p}}^{\xi}, t_{\mathrm{d}}^{\xi})$:

1. $u(\cdot)$ denotes the control used prior to any failure
   $\rightsquigarrow$ u is defined over $[t_{\mathrm{i}}, t_{\mathrm{f}}]$ but implemented over $[t_{\mathrm{i}}, t_{\mathrm{p}}^{\xi}]$
   and corresponds to an **open-loop control**,
2. the control is 0 in $[t_{\mathrm{p}}^{\xi}, t_{\mathrm{p}}^{\xi} + t_{\mathrm{d}}^{\xi}]$,
3. $v^{\xi}(\cdot)$ denotes the control used after the end of the failure
   $\rightsquigarrow$ v^{ξ} is defined over $[t_{\mathrm{p}}^{\xi} + t_{\mathrm{d}}^{\xi}, t_{\mathrm{f}}]$ (if nonempty)
   and corresponds to a **closed-loop strategy $\mathbf{v}$**.

The **satellite dynamics** in the stochastic formulation writes:

$$x^{\xi}(t_{\mathrm{i}}) = x_{\mathrm{i}} \;, \quad \dot{x}^{\xi}(t) = f^{\xi}\big(x^{\xi}(t), u(t), v^{\xi}(t)\big) \;.$$

Stochastic formulation (2)

The problem is to minimize the **expected cost** (fuel consumption)

- w.r.t. the open-loop control u and the closed-loop strategy $\mathbf{v}$,
- the **probability to hit the target** at time t_f being at least p.

$$\min_{u(\cdot)} \mathbb{E}\Big(\min_{v^\xi(\cdot)} K\big(x^\xi(t_\mathrm{f})\big)\Big) \tag{3a}$$

subject to:

$$x^\xi(t_\mathrm{i}) = x_\mathrm{i} \ , \quad \overset{\bullet}{x}{}^\xi(t) = f^\xi\big(x^\xi(t), u(t), v^\xi(t)\big) \ , \tag{3b}$$

$$\|u(t)\| \leq 1 \quad \forall t \in [t_\mathrm{i}, t_\mathrm{f}] \ , \quad \|v^\xi(t)\| \leq 1 \quad \forall t \in [t_\mathrm{p}^\xi + t_\mathrm{d}^\xi, t_\mathrm{f}] \ , \tag{3c}$$

$$\mathbb{P}\Big(C\big(x^\xi(t_\mathrm{f})\big) = 0\Big) \geq p \ . \tag{3d}$$

Probability as an expectation

Let $\mathrm{I}(y) = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$

Then

$$\mathbb{P}\Big(C\big(x^{\xi}(t_{\mathrm{f}})\big) = 0 \Big) = \mathbb{E}\Big(\mathrm{I}\big(\big\| C\big(x^{\xi}(t_{\mathrm{f}})\big) \big\| \big) \Big) \; .$$

Thus, Problem (3) can (shortly) be written:

$$\min_{u(\cdot)} \mathbb{E}\Big(\min_{v^{\xi}(\cdot)} K\big(x^{\xi}(t_{\mathrm{f}})\big) \Big) \tag{4a}$$

$$\text{s.t.} \quad \mathbb{E}\Big(\mathrm{I}\big(\big\| C\big(x^{\xi}(t_{\mathrm{f}})\big) \big\| \big) \Big) \geq p \; . \tag{4b}$$

Formulation (4) opens the possibility to make use of a **stochastic Arrow-Hurwicz algorithm** in order to obtain the solution $u^{\sharp}(\cdot)$.

Missing the target should not contribute to the cost

Whenever a failure occurred, setting $v^\xi \equiv 0$ over $[t_{\mathrm{p}}^\xi + t_{\mathrm{d}}^\xi, t_{\mathrm{f}}]$

- generally makes the satellite misses the target and **does not contribute to the probability constraint**,
- but it forces fuel consumption to its minimum and **contributes to minimize the cost function**.

This yields an **artificially** good expected cost; however, one is not interested in what happens whenever the whole mission has failed.

Therefore, a better formulation is to register only **scenarios** when the target is **effectively hit**; instead of the original expected cost in (4), we thus prefer to deal with a **conditional expected cost**:

$$\mathbb{E}\Big(K\big(x^\xi(t_{\mathrm{f}})\big) \;\Big|\; C\big(x^\xi(t_{\mathrm{f}})\big) = 0\Big) = \frac{\mathbb{E}\Big(K\big(x^\xi(t_{\mathrm{f}})\big) \times \mathrm{I}\big(\big\|C\big(x^\xi(t_{\mathrm{f}})\big)\big\|\big)\Big)}{\mathbb{E}\Big(\mathrm{I}\big(\big\|C\big(x^\xi(t_{\mathrm{f}})\big)\big\|\big)\Big)} .$$

Conditional expected cost

The problem is reformulated as

$$\min_{u(\cdot),\mathbf{v}(\cdot)} \frac{\mathbb{E}\Big(K\big(x^{\xi}(t_{\mathrm{f}})\big) \times \mathrm{I}\big(\big\|C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big)}{\mathbb{E}\Big(\mathrm{I}\big(\big\|C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big)} \tag{5a}$$

$$\text{s.t.} \quad \mathbb{E}\Big(\mathrm{I}\big(\big\|C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big) \geq p \; . \tag{5b}$$

Such a formulation is however not well-suited for the stochastic Arrow-Hurwicz algorithm:

a ratio of expectations is not an expectation!

An useful lemma

Using compact notation, Problem (5) is:

$$\min_{\mathbf{u}} \frac{J(\mathbf{u})}{\Theta(\mathbf{u})} \quad \text{s.t.} \quad \Theta(\mathbf{u}) \geq p \,, \tag{6}$$

in which J and Θ assume positive values.

1. If $\mathbf{u}^\sharp$ is a solution of (6) and if $\Theta(\mathbf{u}^\sharp) = p$, then $\mathbf{u}^\sharp$ is also a solution of

$$\min_{\mathbf{u}} J(\mathbf{u}) \quad \text{s.t.} \quad \Theta(\mathbf{u}) \geq p \,. \tag{7}$$

2. Conversely, if $\mathbf{u}^\sharp$ is a solution of (7), and if an optimal Kuhn-Tucker multiplier $\beta^\sharp$ satisfies the condition

$$\beta^\sharp \geq \frac{J(\mathbf{u}^\sharp)}{\Theta(\mathbf{u}^\sharp)} \,,$$

then $\mathbf{u}^\sharp$ is also a solution of (6).

Back to a cost in expectation

Finally, instead of (5) we aim at solving

$$\min_{u(\cdot),\mathbf{v}(\cdot)} \mathbb{E}\Big(K\big(x^{\xi}(t_{\mathrm{f}})\big) \times \mathrm{I}\big(\big\|C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big) \tag{8a}$$

$$\text{s.t.} \quad \mathbb{E}\Big(\mathrm{I}\big(\big\|C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)\Big) \geq p\,. \tag{8b}$$

The cost function again corresponds to a standard expectation!

Final formulation and associated Lagrangian

$$\min_{u(\cdot)} \left(\mathbb{E}\left(\min_{v^{\xi}(\cdot)} K\left(x^{\xi}(t_{\mathrm{f}})\right) \times \mathrm{I}\left(\left\| C\left(x^{\xi}(t_{\mathrm{f}})\right)\right\|\right)\right)\right)$$

$$\text{s.t.} \quad \mu \rightsquigarrow \quad p - \mathbb{E}\left(\mathrm{I}\left(\left\| C\left(x^{\xi}(t_{\mathrm{f}})\right)\right\|\right)\right) \leq 0 \ .$$

Assuming there exists a saddle point for the associated Lagrangian, we have finally to solve

$$\max_{\mu \geq 0} \ \min_{u(\cdot)} \left\{ \mu\, p + \mathbb{E}\left(\min_{v^{\xi}(\cdot)} \left(K\left(x^{\xi}(t_{\mathrm{f}})\right) - \mu\right) \times I\left(\left\| C\left(x^{\xi}(t_{\mathrm{f}})\right)\right\|\right)\right)\right\} .$$

Remark: it is convenient to put apart the **no-failure event**, namely $\left\{\mathbf{t}_{\mathrm{p}} \geq t_{\mathrm{f}}\right\}$ in the problem formulation. For the sake of simplicity, we do not present this last problem improvement. We will denote by π_{f} the probability of this event.

Algorithm overview (1)

In order to solve

$$\max_{\mu \geq 0} \ \min_{u(\cdot)} \left\{ \mu \, p + \mathbb{E}\Big(\underbrace{\min_{v^{\xi}(\cdot)} \big(K\big(x^{\xi}(t_{\mathrm{f}})\big) - \mu\big) \times \mathrm{I}\big(\big\| C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big)}_{W(u,\mu,\xi)} \Big) \right\} .$$

that is,

$$\max_{\mu \geq 0} \ \min_{u(\cdot)} \Big\{ \mu \, p \ + \ \mathbb{E}\big(W(u,\mu,\xi) \big) \Big\} ,$$

we use a **stochastic Arrow-Hurwicz algorithm** (see [2] and [3]).

Algorithm overview (2)

Arrow-Hurwicz algorithm

At iteration k,

1. draw a failure $\xi^k = (t_{\mathrm{p}}^{\xi^k}, t_{\mathrm{d}}^{\xi^k})$ according to its probability law,
2. update $u(\cdot)$:

$$u^{k+1} = \Pi_{\mathcal{B}}\Big(u^k - \varepsilon^k \, \nabla_u W(u^k, \mu^k, \xi^k)\Big) \ ,$$

3. update μ:

$$\mu^{k+1} = \max\Big(0, \mu^k + \rho^k \big(p + \nabla_\mu W(u^{k+1}, \mu^k, \xi^k)\big)\Big) \ .$$

Downstream closed-loop problem

Consider the inner optimization problem:

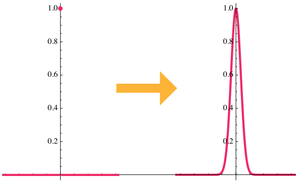
$$W(u,\xi,\mu) = \min_{v^{\xi}(\cdot)} \Big\{ \big(K\big(x^{\xi}(t_{\mathrm{f}})\big) - \mu\big) \times \mathrm{I}\big(\big\| C\big(x^{\xi}(t_{\mathrm{f}})\big)\big\|\big) \Big\} ,$$

- If reaching the target is **impossible** ($C\big(x^{\xi}(t_{\mathrm{f}})\big) \neq 0 \ \ \forall v^{\xi}$), the best thing to do is to **stop immediately**: $W \equiv 0$.
- If reaching the target is **possible** but **too expensive** (that is $K\big(x^{\xi}(t_{\mathrm{f}})\big) \geq \mu$), the best thing to do is again to **stop immediately**: $W \equiv 0$.
- Therefore, it must be checked that one can reach the target while maintaining fuel consumption below a threshold, and the corresponding optimal control $v^{\xi}_{*}(\cdot)$ must be computed.

At every iteration k, we must **evaluate** function W as well as **its derivatives** w.r.t. $u(.)$ and μ. But W is not differentiable!

First difficulty: I is not a smooth function

$$\mathrm{I}(y) = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{otherwise,} \end{cases} \quad \rightsquigarrow \quad \mathrm{I}_r(y) = \begin{cases} \left(1 - \frac{y^2}{r^2}\right)^2 & \text{if } y \in [-r, r], \\ 0 & \text{otherwise.} \end{cases}$$



There are specific rules to drive r^k to 0 as the iteration number k goes to infinity in order to obtain the best asymptotic **Mean Quadratic Error** of the gradient estimates (see [1]).

Second difficulty: solving the approximated problem

The approximated closed-loop problem to solve at each iteration is:

$$W_r(u^k, \xi^k, \mu^k) = \min_{v^\xi(\cdot)} \left\{ \left(K\left(x^\xi(t_{\mathrm{f}})\right) - \mu^k \right) \times \mathrm{I}_r\left(\left\| C\left(x^\xi(t_{\mathrm{f}})\right) \right\| \right) \right\} .$$

In this setting, we have to check if the target is reached up to r.

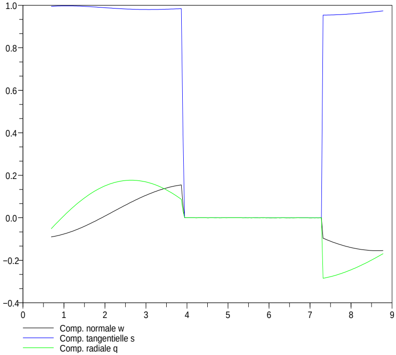
In practice, the solution of the approximated problem is derived from the resolution of two standard optimal control problems.

1. Problem formulation
2. Modeling improvement
3. Stochastic Arrow-Hurwicz algorithm
4. Numerical results
   - Brief description
   - Results for different values of p
   - The price of safety

Mission description

Interplanetary mission (Earth-Mars trajectory):

- duration of the mission: 450 days,
- $\mathbf{t}_{\mathrm{p}}$: exponential distribution s.t. $\mathbb{P}\big(\mathbf{t}_{\mathrm{p}} \geq t_{\mathrm{f}}\big) = \pi_{\mathrm{f}} \approx 0.58$,
- $\mathbf{t}_{\mathrm{d}}$: exponential distribution s.t. $\mathbb{P}\big(2 \leq \mathbf{t}_{\mathrm{d}} \leq 7\big) \approx 0.80$.



Using normalized units:

- $t_{\mathrm{i}} = 0.69$ and $t_{\mathrm{f}} = 8.73$.

The **deterministic optimal control** has a "bang–off–bang" shape. Along the **deterministic optimal path**, the probability to recover a failure is: $p^{\mathrm{det}} \approx 0.94$.

Parameters tuning

Gradient step length:

$$\varepsilon^k = \frac{a}{b + k} \quad , \qquad \rho^k = \frac{c}{d + k} \, ,$$

$\rightsquigarrow$ usual for a stochastic gradient algorithm.

Smoothing parameter:

$$r^k = \frac{\alpha}{\beta + k^{\frac{1}{3}}} \, ,$$

$\rightsquigarrow$ MQE reduced by a factor 2000 in about 100.000 iterations.

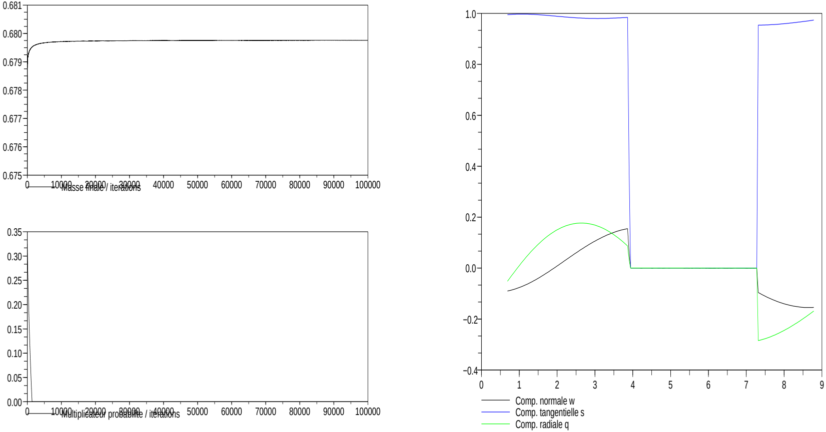


Figure: Probability level $p < \pi_{\mathrm{f}}$

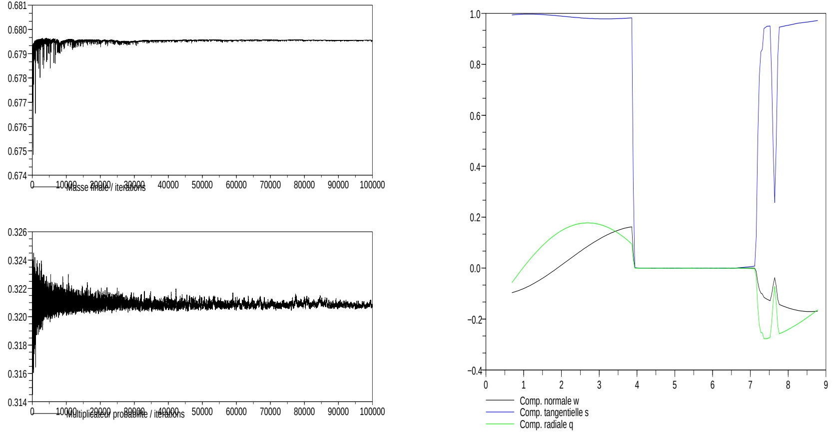


Figure: Probability level $p = 0.750$

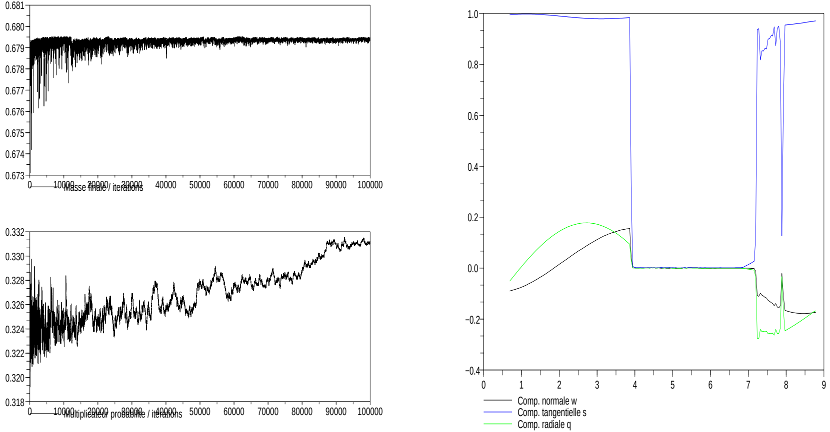
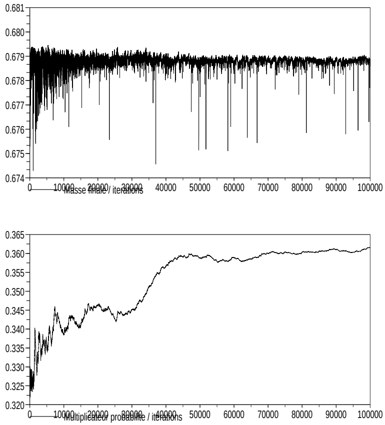


Figure: Probability level $p = 0.960$



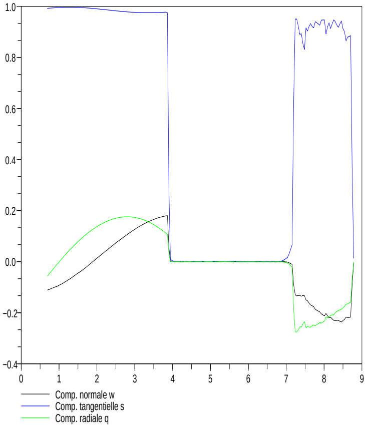
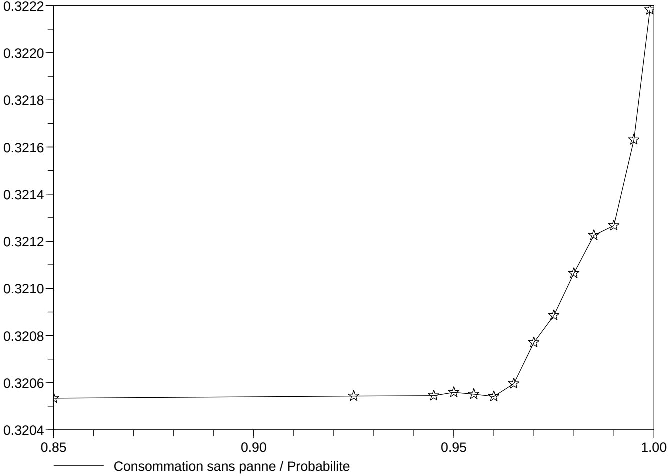


Figure: Probability level $p = 0.990$

Fuel consumption versus probability level



Conclusion

Main conclusion

We are able to deal with probability constraints in the optimal control framework.

Future works

- *From the theoretical point of view*:
  - existence of a saddle point for the constrained problem,
  - smoothing process (results available only for inequality constraints).
- *From the numerical point of view*:
  - efficient solver for the downstream problem,
  - computer parallelization.

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Thank you for your attention. Any question?